Making connections and seeking understanding:
Mathematical tasks in English, French and German textbooks

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Abstract
After a thorough literature review we contend that connectivity and the making of 'connections' can be used as an analytical tool to analyse tasks with respect to potential pupil understanding of mathematics, and we subsequently identify ways of making connections in mathematical tasks in textbooks. To exemplify the points made we examined mathematical tasks in selected English, French and German lower secondary textbooks (and their treatment of 'Negative Numbers'). Connections are explored in terms of conceptual understanding and interconnectedness of mathematical content knowledge, amongst other things, and tasks and exercises are investigated with respect to cognitive demand and contextualisation. An analysis of the data suggests that tasks may provide different connections within mathematics and beyond mathematics for students in school textbooks. We argue that particular tasks provide different (mis)representations of mathematics for their students in school textbooks, in particular with respect to connectivity. Despite perhaps gaining proficiency at certain kinds of procedures and tasks, some students may well have gained at best a fragmented sense of the mathematics and understood few if any connections that tie together the procedures they had studied. It can be argued that through these disconnected activities students are likely to develop perspectives on mathematics that may impede them in their use and acquisition of other mathematical knowledge. If we can assume that learning with understanding is enhanced by making connections, mathematical tasks should reflect this. They are likely to influence students' perception of what mathematics is and what it is to behave mathematically.

Introduction

During the past decades there has been much concern and discussion about student conceptual understanding and their expertise in terms of mathematical thinking, reasoning and problem-solving (for example National Research Council, 1989; Hiebert and Carpenter, 1992). The underlying goals for addressing these concerns have been to enhance student understanding of mathematics and to help them to develop their capacities to move beyond procedural knowledge to 'think mathematically'. Hiebert et al (1997) argue that classrooms that facilitate mathematical understanding share some core common features. Their framework consists of five dimensions that are said to work together to shape classrooms into 'particular kinds of learning environments': (a) the nature of the learning tasks; (b) the role of the teacher; (c) the social culture of the classroom; (d) the kind of mathematical tools that are available; and (e) the accessibility of mathematics for every student (p.2). In this paper we concern ourselves with the 'nature of the learning tasks' (Hiebert et al, 1997).

Furthermore, it is important that students have 'frequent opportunities to engage in dynamic mathematical activity that is grounded in rich, worthwhile mathematical tasks' (Henningsen & Stein, 1997, p.525), and it is argued that this is an essential component for understanding in order for connections to be made by the learner (Hiebert et al, 1997).

It is reasonable to argue that materials, such as textbooks and worksheets for example, are an important part of the context in which pupils and teachers work. In recognition of the central importance of such documents, the framework for the Third International Mathematics and Science Study (TIMSS) included large-scale cross-national analyses of mathematics curricula and textbooks as part of its examination of mathematics education and attainment in almost 50 nations. Concerns have been expressed about the quality of textbooks, for example, and about their persuasive influence, and it appears that the textbook content, and how it is used, is a significant influence on students' opportunity to learn and their subsequent achievement (Robitaille and Travers, 1992). It is also commonly assumed that textbooks are one of the main sources for the content covered and the pedagogical styles used in classrooms (Valverde et al, 2002). Teachers often rely heavily on textbooks in their day-to-day teaching, and they decide what to teach, how to teach it, and the kinds of tasks and exercises to assign to their students.

Considering this fact, it is sensible to analyse mathematical tasks in textbooks to exemplify the issues raised.

Thus, of central importance in this article is the concept of 'academic' or 'mathematical' task (Doyle, 1986, 1988), a concept which is used here as an analytical tool for examining subject matter as a classroom process rather than simply as a context variable in the study of learning. The purpose of this paper is to develop an analytical tool for analysing tasks in mathematics textbooks. This is done by giving a general overview of research in the areas of textbooks internationally in relation to mathematical tasks, and subsequently to develop from that literature a framework for the analysis of textbook tasks. From this, we look at particular examples from textbooks to exemplify the elements of the framework.

Textbooks and the importance of mathematical tasks

It appears that textbooks, for better or worse, define and represent the subject for very many students, and they influence how those students experience mathematics. Textbooks provide children with opportunities to learn, and learn those things which are regarded as important by their government. Teachers mediate textbooks by choosing and affecting tasks, and in that sense student learning, by devising and structuring student work from textbooks.

It has been shown that teachers use textbooks heavily for their selection of tasks (Kuhs and Freeman, 1979; Luke et al, 1989; Pepin and Haggarty, 2001). Furthermore, Doyle (1988) argues that the tasks teachers assign to students influence to a large extent how students come to understand the curriculum domain. Moreover, in his opinion, they serve as a context for student thinking not only during, but also after instruction. This premises that tasks, most likely chosen from textbooks, influence to a large extent how students think about mathematics and come to understand its meaning. Indeed, Henningsen and Stein (1997) argue that

> the tasks in which students engage provide the contexts in which they learn to think about subject matter, and different tasks may place different cognitive demands on students …. Thus, the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of their subject matter with which they are engaged. Students develop their sense of what it means to "do mathematics" from their actual experiences with mathematics, and their primary opportunities to experience mathematics as a discipline are seated in the classroom activities in which they engage … (p.525)

Hiebert et al (1997) similarly argue that students

> also form their perceptions of what a subject is all about from the kinds of tasks they do. … Students' perceptions of the subject are built from the kind of work they do, not from the exhortations of the teacher. … The tasks are critical.(p.17/18).

In order to study tasks, it is necessary to examine the concept of "task" and subsequently select what is relevant for the mathematics classroom in terms of textbooks. Doyle (1988, also 1983, 1986) defines academic tasks in terms of four components. He calls attention to four aspects of work in class:

(a) the end product to be achieved;
(b) a set of conditions and resources available to accomplish the task;
(c) the operations involved to reach the goal state;
(d) the importance of the task.
(Doyle, 1988, p.169)

He also points to the problem that a task exists at several different levels at once (as announced by the teacher, or as interpreted by the student, for example). This multiplicity of meanings, clearly likely to complicate task research, alerts us to the idea that curriculum content can be represented in a variety of fundamentally different ways in the classroom. For example, if one looks simply at the level of cognitive mathematical content demand in each tasks, one may fail to recognise that the task itself might require the student to calculate answers to a well-structured exercise at one extreme, whereas at the other extreme, require the application of conceptual understanding to a 'real-world' problem (Doyle, 1988).

Thus, the way in which the student is being asked to work with the content determines to some extent the cognitive demand of the task.

In his research on mathematics instruction Schoenfeld (1988) argues that students, despite gaining proficiency at certain mathematical procedures through the completion of tasks, gained a fragmented sense of the subject matter and understood few connections that tie together the procedures they had studied. Indeed, he argues that most textbooks present ‘problems’ that can be solved without thinking about the underlying mathematics but by ‘blindly applying’ the procedures that have just been studied (p163). Moreover, he claims that the students “developed perspectives regarding the nature of mathematics that were not only inaccurate, but were likely to impede their acquisition and use of mathematical knowledge” (p.145). We explore the idea of connectedness in a later section.

The analysis of tasks

In this section we will review the literature in terms of the different ways in which tasks could be analysed. These include general features of mathematical tasks to enhance learning; the notions of ‘appropriateness’ of tasks and ‘respectful’ tasks; cognitive demand in mathematical tasks; the contextual features and purposes of tasks; and, finally, connectedness and mathematical knowledge.

(1) General features of mathematical tasks to enhance learning

Kilpatrick et al (2001) give a comprehensive view of what they regard as successful mathematics learning. They coin the term “mathematical proficiency” to capture what they think it means for anyone to learn mathematics successfully. In their view, mathematical proficiency has five “interwoven” and “interdependent” strands.

- *conceptual understanding* - comprehension of mathematical concepts, operations, and relations
- *procedural fluency* - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence* - ability to formulate, represent, and solve mathematical problems
- *adaptive reasoning* - capacity for logical thought, reflection, explanation, and justification
- *productive disposition* - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

(Kilpatrick et al, 2001, p.5)

In terms of instruction, Kilpatrick et al (2001) argue that the quality of instruction depends, for example, on the tasks selected for instruction and their cognitive demand. They further claim that the teacher’s expectations about the mathematics particular students are able to learn can “powerfully influence the tasks the teacher poses for the students, the questions they are asked,…in other words, their opportunities and motivation for learning.” (Kilpatrick et al, 2001, p.9). This supports findings of Haggarty and Pepin (2002) in English, French and German classrooms, where teacher expectation influenced their choice and presentation of tasks to students.

Hiebert et al (1997) reflect explicitly on the nature of tasks that may build mathematical understanding. They believe that students build mathematical understanding by ‘reflecting and communicating’, and tasks should allow and encourage these processes. This means that such tasks should have the following features:

> First, the tasks must allow the students to treat the situations as problematic, as something they need to think about rather than as a prescription they need to follow. Second, what is problematic about the task should be the mathematics rather than other aspects of the situation. Finally, in order for students to work seriously on the task, it must offer students the chance to use skills and knowledge they already possess. Tasks that fit these criteria are tasks that can leave behind something of mathematical value for students. (p.18)

Interestingly, Hiebert et al (1997) use the term 'appropriate' for tasks which have the above mentioned characteristics. In the next section we discuss 'appropriate tasks' as seen by other members of the education community.

(2) Cognitive demand

Smith and Stein's work (1998) is useful in terms of positioning the nature of tasks. They argue that the nature of the task determines what students learn. They also use four hierarchical categories of cognitive demand to assess tasks:

- Memorisation
- Procedures without connections to concept or meaning
- Procedures with connections to concept or meaning
- Doing mathematics

In another study Stein et al (1996) query whether tasks that start out as cognitively demanding may decline into less demanding activities over the course of the lesson. High level tasks are often perceived by students as less structured, more complex, and longer than they are normally exposed to. According to research by Doyle (1988) and others, students often perceive such tasks as ambiguous: they do not know what to do and how. In turn they often urge teachers to explain such tasks, thereby potentially eliminating some of the complexity and perhaps even cognitive demand of the task.

However, Smith and Stein (1998) point out that although the decision to use a cognitively demanding task does not necessarily lead to high-level engagement by students,

> Starting with a good task does, however, appear to be a necessary condition [for high-level engagement], since low-level tasks almost never result in high-level engagement (p344)

Nevertheless, Nicely (1985) asserts that mathematics textbooks have "a poor track record" in terms of higher order thinking skills (p.26).

(3) Contextual features and purposes of tasks

The research literature shows increasing evidence of how mathematics is used in everyday activities (e.g. Carraher et al, 1985, 1987; Lave 1988). Assuming the pervasiveness of mathematics embedded in everyday activities, and the motivation it stimulates to 'get something done', an important issue for textbook task analysis is to what extent, and in which ways, these 'real world' experiences are incorporated. In terms of learning, the argument is likely to centre around the idea that mathematical knowledge and understanding would be enhanced and become more coherent for learners if they could establish connections between the 'networks' of out-of-school experiences and those of in-school mathematics, and these should become integrated. Problem-solving in one setting should be informed by strategies learnt in other settings.

Skovsmose (2002) talks about 'learning milieus' and distinguishes between three different 'paradigms of exercises': those with references to pure mathematics (context unembedded); those with reference to a semi-reality (context embedded tasks, albeit contracted realities); and 'real-life references' (Skovsmose, 2002, p.119). Whilst his notion of semi-reality can be seen to have features similar to very many textbook tasks, the point Skovsmose makes about the task is that such a situation is artificial and is unlikely to lead to students 'awareness of and willingness to draw on their own intellectual capabilities when making mathematical decisions and judgements' (Cobb and Yackel, 1998, p170)

It is beyond the scope of this paper to examine the complexity of pupil learning relating to working with tasks in context, but it is useful to comment nevertheless that a student's mathematical procedure and performance can be largely determined by the particular context used in a task, so that students interact with the context of a task in many different and unexpected ways and this interaction is, by its nature, individual (Boaler, 1993). Thus, it would be unhelpful to suggest that the inclusion of tasks in context necessarily leads to understanding.

(4) Connections, connectedness and mathematical knowledge

'Connectedness' is not a new idea in the field of education, or indeed in mathematics education. Hiebert and Carpenter (1992), for example, believe that it is essential to make connections in mathematics if one intends to develop mathematical understanding. They emphasise the importance of learning with and for understanding. According to them, understanding can be defined as:

> …the way information is represented and structured. A mathematical idea or fact is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of the connections. (p67).

For them, what is essential for facilitating student understanding involves a number of principles, amongst them that understanding can be characterised by the kinds of relationships or connections that have been constructed between ideas, facts, procedures, for example. They describe understanding in terms of the way an individual's internal representations are structured and connected, and also how these internal representations are structured and connected to external representations. These representations would include spoken language, or written symbols, or analogies. Thus, a mathematical idea or procedure or fact is understood if it is linked to existing networks with strong and numerous connections. This means that understanding is not an all or nothing phenomenon (Hiebert and Carpenter, 1992) and internal networks can be thought of as dynamic. It also emphasises the importance of past experiences for interpreting and understanding new experiences. As Ma (1999) argues,

> When it is composed of well-developed, interconnected knowledge packages, mathematical knowledge forms a network solidly supported by the structure of the subject (p120)

Hiebert and Carpenter (ibid) further point out that if mathematical tasks are overly restrictive, students' internal representations are severely constrained, and the networks they build are bounded by these constraints (p76). Further, the likelihood of transfer across settings becomes even more problematic (p79). Indeed Hiebert (1986) commented that many students did not connect the mathematical skills they possessed with the symbols and rules taught in school:

> I shall argue that it is the *absence* of these connections that induces the shift from intuitive and meaningful problem-solving approaches to mechanical and meaningless ones (Hiebert, 1984, p498)

This in turn reminds the reader of ideas relating to instrumental and relational understanding (Skemp, 1976), with instrumental 'understanding' relating to 'rules without reasons', which we might understand here as rules without connections. It is suggested that when students assumed instrumental beliefs about mathematics they failed to develop genuine understanding: the connections simply were not there to be made.

Whilst it could be argued that teachers might mediate texts in such a way that they support students in making connections, it is by no means clear that they would (for example Schoenfeld, 1988, Haggarty and Pepin, 2002) or, indeed, that they could. For example, in terms of international and comparative research, Ma (1999) compared Chinese and US elementary teachers' mathematical knowledge. She found that Chinese elementary teachers perceived mathematical concepts as interconnected, which was in contrast to US colleagues who perceived these concepts as arbitrary collections of facts and rules. She developed a notion of 'profound understanding of fundamental mathematics' (PUFM), an argument for structured, connected and coherent knowledge (Ball et al, 2001), which is 'deep', 'broad' and 'thorough' (Ma, 1999, p 120) and this was seen as one of the factors for student enhanced mathematical performance.

Our analysis of the literature has alerted us, therefore, to the need for textbooks to make connections explicit and to support the making of connection through multiple representations. Further, we have argued that as part of an analysis of textbooks, tasks in those textbooks should be scrutinised according to the extent to which they:

- emphasise relational rather than procedural or instrumental understanding
- make connections with what students already know
- make connections with the underlying concepts and relations being learnt
- make connections in terms of linking and carrying out procedures accurately and appropriately
- make connections within mathematics and across other subjects
- are embedded in contexts which help to make connections with 'real life'
- make high cognitive demand on pupils
- connect different representations (analogies, worked examples)

The Study

In a previous study (Pepin and Haggarty, 2001; Haggarty and Pepin, 2002) we used an analysis schedule developed from the literature to investigate textbooks, and subsequently linked this to teachers' use of those textbooks in English, French and German lower secondary mathematics classrooms. This gave us an understanding of the similarities and differences of mathematics textbooks and how these were influenced by educational traditions in the three countries.

Our growing interest in the research literature on 'mathematical understanding' and 'connections', and in relation to this on the mathematical tasks offered in textbooks, encouraged us to re-analyse the textbooks from that study on the basis of that new research literature. Thus, in this study we used our knowledge of textbooks and the analyses of textbooks to develop a deeper understanding of connections made in textbook tasks. Textbooks which were originally identified (in 2000) as amongst the ones most frequently purchased for years 7 (6è*me*, *Jahrgang 6*), 8 (*5ème*, *Jahrgang 7*) and 9 (*4ème*, *Jahrgang 8*)[1] and still used in classrooms in the three countries were chosen for re-analysis. The topic of 'directed numbers' was selected for a more detailed analysis, because this topic was regarded as relatively self-contained and likely to be taught as a new topic, particularly in years 7 and 8. There was also reference to years 9 and 10 in terms of follow-up of topics and coherence through the years.

In order to exemplify the earlier points elicited from the literature, individual tasks were analysed with respect to

1. familiar situations- tasks which make connections with what students already know;
2. context embeddedness- 'real life' tasks that make mathematics useful and worthwhile for students (for example, in introductory tasks/activities);
3. conceptual understanding - tasks which make connections with the underlying concepts and relations being learnt;
4. procedural fluency- making connections in terms of linking and carrying our procedures accurately and appropriately;
5. cognitive demand- tasks which ask for justification and generalisations;
6. connections within mathematics and across other subjects;

[1] The following textbooks were chosen for re-analysis
- Germany: Lambacher-Schweizer (*Gymnasium*); Einblicke Mathematik (*Hauptschule*);
- France: Cinq sur Cinq;
- England: Keymaths.

7. mathematical representations- tasks which connect different representations.

Results and findings

From our knowledge of mathematics textbooks and tasks in English, French and German textbooks, and applying our analytical tool, we identified tasks that reflected the seven criteria particularly well or particularly badly. We thus did not attempt to compare tasks according to particular criteria, but rather highlight particular examples, or counter examples, that illustrate the points made.

1. Tasks which make connections with what students already know

All the textbooks used familiar analogies of, for example, temperature (predominantly); lifts; quiz shows; heights above and below sea-level; financial gain and loss. In the English KM 7^2 (Key Maths 7^2) and French 5 sur 5 (6ème) textbooks, the analogy was usually simply drawn on without helping the student to engage with it. In the German LS7 (Lambacher Schweizer 7) textbook, however, students were provided with an explanation to help them engage with the ideas.

In this particular textbook (Lambacher Schweizer 7- LS7) students were first given the opportunity to read a small newspaper snippet about 'red numbers' and short-term labour, reasoning that the company's income decreased. Students were subsequently asked what "red numbers" meant in this newspaper snippet. Further down the page it was explained that in the past in accountancy and in account statements black numbers used to represent assets, and red numbers debts. The diagram further down the page shows the gains of a company over a whole year. In the months of June until August "red numbers" were registered, that is instead of gains they had losses. (LS7, p.78) On the subsequent page pupils are told in an exercise:

> Represent in a diagram the gains and losses of a company:
>
> | | |
> |---|---|
> | 1st quarter (of the year) | 80 000 DM gains |
> | 2nd quarter | 150 000 DM losses |
> | 3rd quarter | 40 000 DM gains |
> | 4th quarter | 40 000 losses. |
>
> (Question 7, p.79)

So, although it is important to include familiar situations, it seemed the connection could be strengthened by providing bridging ideas.

2. 'Real Life' tasks that make mathematics useful and worthwhile for students

The textbooks put different emphases on context embeddedness in different countries. Whereas in English and German textbooks approximately half of all tasks were context embedded, in French books this was reduced to approximately a third. This means that for any German, French or English student at least every second task is situated totally in the abstract world of mathematics, thus not likely to connect to the world students live in. Of course this begs the question of the extent to which the remaining context embedded tasks help student appreciation of mathematical ideas and thinking, but this goes beyond the scope of this paper.

All countries used 'real life' tasks, but they did not necessarily make the mathematics useful, nor did their solution necessarily draw on the relevant mathematics in order to be solved. Referring to Skovsmose's real-life examples of tasks (Skovsmose, 2002), Skovsmose talks about 'learning milieus' and distinguishes between three different 'paradigms of exercises': those with references to pure mathematics (context unembedded); those with reference to a semi-reality (context embedded tasks, albeit contracted realities); and 'real-life references' (Skovsmose, 2002, p.119). Considering that real-life exercises and tasks might give students meaningful contexts and connections to familiar experiences, it is interesting that there are so few in any of the textbooks we analysed and we found that most tasks were in the milieu of semi-reality.

To exemplify this aspect we have identified two similar tasks from the English KM 7^2 and the French 5 sur 5 (6ème) textbooks. In the case of early work on negative numbers KM 7^2, questions are all asked in context - usually that of temperature – but also of lifts in buildings, quiz shows and altitudes, for example. Many of these contexts are suggested in the Key Stage 3 National Strategy for mathematics (2001). However, because there was no commentary or attempt to link the questions to gain some generalised understanding of negative numbers, each question was likely to be tackled in isolation and could be answered using common-sense techniques. Thus there was the real danger that negative numbers were in danger of remaining disconnected from mathematics. An example in KM 7^2 (p 250) textbook is given in the following:

> Here is the control panel in a lift.
>
> | | |
> |---|---|
> | 4 | Fourth floor |
> | 3 | Third floor |
> | 2 | Second floor |
> | 1 | First floor |
> | 0 | Ground floor |
> | -1 | Underground car-park |
>
> a) what number is used for the ground floor?
> b) Where do you go if you press -1 ?
> c) You go from the first floor to the fourth floor. How many floors do you go up?
> d) You go from the third floor to the car-park. How many floors do you go down?
>
> (KM 7^2, p.250, Question 1, ex. 11.3)

A task with the same context was found in the '5 sur 5' (6ème). We read:

> A tower consists of 12 upper levels and 4 levels below ground. What do you imagine the lift board will look like?
>
> (p.218, Question 2)

This task was illustrated with a cartoon where a man waits with two heavy suitcases and a backpack in front of the lift, which is 'out of order'.

Interestingly, and in both examples, it appears that the underlying ideas about negative numbers, particularly in the first of these examples, are subordinated to the context, and the context largely points pupils towards a context-bound and common-sense solution. Here, the context hides the fact that it relates to negative numbers, and that the context-bound questions neither relate to each other nor to the underlying mathematical ideas of negative numbers.

3. Tasks which make connections with the underlying concepts and relations

Students may have problems with negative numbers because they apply what they know (in terms of concepts) about natural numbers to negative numbers. Research literature (e.g. Vlassis, 2004; Gallardo, 2002) argues that students attempt to reconcile their arithmetic presuppositions about natural numbers and the algebraic rules required to operate with negatives. Relating the different sets of numbers may help students in making connections, but they may also 'produce' overgeneralisations, i.e. that students think that all rules are 'transferable' and interchangeable between different sets of numbers.

Studying conceptual change about negatives it must be born in mind that

> these numbers do not only constitute a difficulty in themselves since they represent 'fictive numbers' (Glaeser, 1981), but, used in operations … they modify the role of the minus sign which becomes not only an 'operating' sign but also a 'predictive' sign (Glaeser, 1981). Based on students' difficulties that are reported in the literature above, we suggest that the minus sign

> plays a major role in the development of understanding and using negative numbers.
>
> (Vlassis, 2004, p.471)

According to Vlassis (2004) the concepts underpinning negative numbers are related to the different uses of the minus sign:

- Unary function- 'predictive'- minus sign as 'structural signifier'- integers
- Binary function- minus sign as the 'operating' sign, taking away, the difference between two numbers, movement on the number line;
- Symmetric function- taking the opposite of, inversion

The third of these was not addressed in KM 7^2 .

In the LS7, and under the topic of negative numbers, we found more examples of tasks that made connections to underlying concepts than in any other textbook. These were generally given at the end of each subchapter, and when the procedural tasks - and there were many of them - had been completed.

As examples of useful tasks, we have chosen the following, because we believe that the exercise uses some effective questioning to help students to get close to the different conceptual notions within the topic of negative numbers.

> Which of the following statements are true, which are false?
> (a) The sum of a rational number and its opposite is always zero.
> (b) The difference of a rational number and its opposite is always zero.
> (c) The sum of two rational numbers can always be written as the sum or difference of their moduli.
> (d) The difference of two rational numbers can always be written as the sum or difference of the moduli of those numbers.
>
> (LS7, p.93, Question 10)

> (a) With which number do you have to multiply a number, in order to get its opposite number?
> (b) Which number do you have to add to a number, in order to get its opposite number?
>
> (LS7, p.99, Question 15)

Whereas in the first case, the conceptual base of rational numbers is explicitly addressed, in the second case it is implicit. However, in both cases the concept of 'opposite' number is addressed.

We also found tasks where the 'context-embeddedness' appeared to mask the underlying concept, indeed the concept seemed to become subordinated to the context itself. This was sometimes compounded when the solution itself did not require the student to answer the question by drawing on the concept. For example, in the KM 7^2 students were asked:

> (a) Amy loses her coat and must pay £55 for a new one. She has £26 saved and she earns £15. Amy borrows the rest from her mother. How much does she borrow?
> (b) Amy's aunt gives Amy £20 for her birthday. Amy pays her mother back. How much does Amy have left?"
>
> (KM 7^2, p250, Question 5)

4. Making connections in terms of linking and carrying our procedures accurately and appropriately

The LS7 textbook offered many opportunities for students to achieve procedural fluency throughout the chapter of negative numbers. For example, there were 76 questions aimed at procedural fluency in the exercise part of a subchapter on ‘Calculating with opposites’ (LS7, p.92-93). On page 93 we find questions such as:

- Find x by searching for the opposite: 27+x=0 (qu 3a)
- Write the expressions without brackets and then calculate: (+79) – (+85) (qu 4b)
- (-63.2)- (-98.4) (qu 5c)
- (-5/4) – (-2/3) (qu 6d)
- Write as a sum and calculate: Calculate – 78 – 95 (qu 7d); 19-(-5) (qu 7e)
- Calculate: 2.05-5.64 (qu 8f)

The ‘5 sur 5’ (6ème) also offered a variety of activities for achieving procedural fluency (see figure 1).

Fig 1 about here

5. Tasks which ask for justifications and generalisations

We found few examples of high cognitive demand. However, amongst the most demanding we found the following from the LS7:

> A ‘snail’ starts at A (-1/0) and winds itself from there around the origin O. The point B has the distance 5/4 from 0, C the distance (5/4) squared, etc.

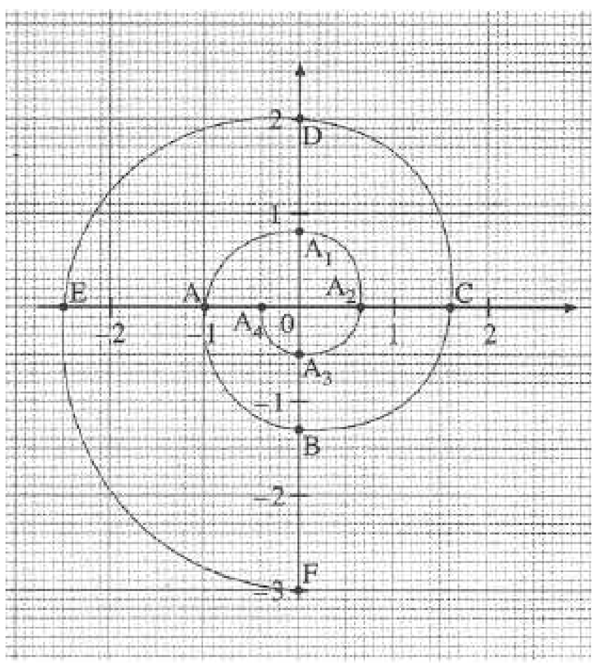

Fig. 2

(a) Find the co-ordinates of the points B, C, D, E, F, G and H.
(b) Imagine the snail continuing its journey towards the inside. It then ‘cuts’ the y-axis in A1, the x-axis in A2 etc. Find the co-ordinates of A1, A2, A3, A4 etc.

(LS7, p.102, Question 16)

It must be born in mind that in Germany all aspects of and operations with negative numbers are covered in Year 8 (probably over two or three months), whereas in England the topic is started in Year 7 (and revisited in subsequent years). In France there is clear distinction/progression from year 7 to Year

9, where negative numbers are introduced in Year 7, addition/subtraction addressed in Year 8, and multiplication/division in year 9- and this is reflected in the textbooks.

We also found examples where the cognitive demand could have been increased but the opportunity was not taken. For example, in the KM 7² students were given the following question:

> James subtracts one number from another on his calculator and gets the answer 8. What would his answer be if he did the subtraction with the second number first?
>
> (KM 7², p.250, Question 4)

In this example the context itself seems to delimit what might be explored here. Indeed, this was the only example from the whole section in this particular textbook which offered possibilities for exploration and generalisation. Instead, we are left with a cognitively undemanding task and, as Smith and Stein argue (1998), 'starting with a good task does, however, appear to be a necessary condition [for high level engagement], since low-level tasks almost never result in high-level engagement' (p.344).

6. Connections within mathematics and across other subjects

We were able to find tasks where connections were made within mathematics and across other subjects. For example, in the LS7 we found a question in which negative numbers were related to other mathematical concepts, such as mean and deviation from the mean.

> Five children of a family have the following weights at birth:
> 3345g; 3185g; 3320g; 3265g; 3280g.
> Calculate the deviation of the weights at birth from the mean weight of the five children. Represent the deviations on a scale.
>
> (LS7, p.79, Question 9)

Similarly, in the '5 sur 5' (6ème) we found a task entitled "The ice melts!"

> When heating up/melting ice cubes the temperature is increased every minute and the following graph is obtained:

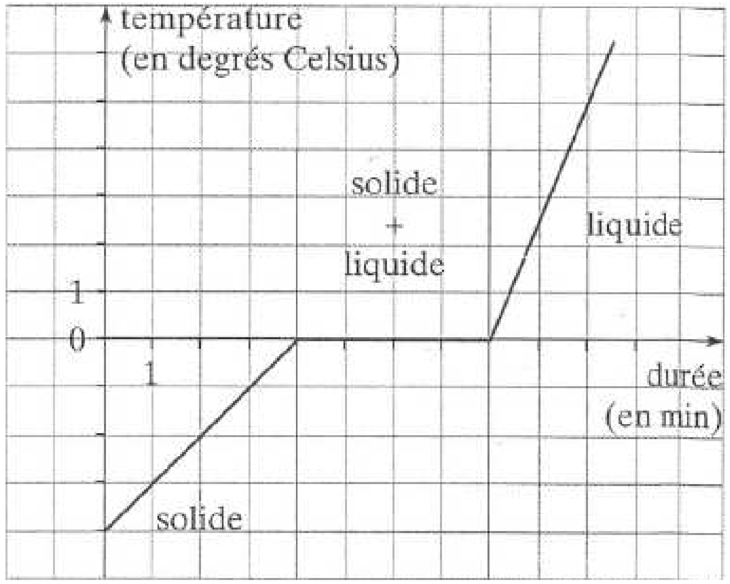

Fig. 3

1. What is the temperature of the ice cubes at the start?
2. What is the temperature after three minutes, five minutes, ten minutes?
3. After how much time do you get a liquid?

(5 sur 5, 6è, p.221, Question 29)

7. Tasks which connect different representations

In LS7, negative numbers were first introduced by explaining their use in contexts, such as thermometer; water height. For this, a pictorial representation was provided, e.g. a thermometer with a reading of 10 degrees. It was explained that heights of mountains, for example, are measured in 'height above sea level', and an illustration was provided.

Further down the page it was explained that "temperature, water height, geographical heights" may have values below zero, and these can be "graphically represented" on a number line (which is pictured). "The black numbers 1,2,3, … and the red numbers are reflections with respect to zero." (p.78)

What is clear from this introductory page is that an attempt has been made to start with contexts in which the mathematical concept can be embedded. Contexts are used that students may already know. Moreover, these contexts are then related to another representation, the number line. This resonates with research in mathematics education emphasising the importance of helping students build connections among multiple representations of a problem or concept, and of helping students link the mathematics to solutions based on experience with familiar situations (e.g. Grouws, 1992; Hiebert, 1986).

Furthermore, in the subsequent subchapter the analogy of temperature is picked up again, and so is the concept of the number line, and addition and subtraction of negative numbers is explained on the number line. The idea of symmetry (reflection about 0) is also developed, and the reader is introduced to 'opposite numbers' (see above) (i.e. -5 is the opposite number of 5, and 5 is the opposite number of -5). Interestingly, it is explained that

> if one extends the natural numbers to their opposites, one gets the set of whole numbers, that is the numbers …; -3; -2; -1; 0; 1; 2; 3; … Similarly, if one extends the fractions to their opposites, one gets the set of rational numbers."

(LS7, p.80)

This is then exemplified, in worked examples, with whole numbers and fractions, and also in all four quadrants of the co-ordinate system.

To follow through the use of the number line, in the third subchapter (p.82-83) rational numbers are ordered, that is organised on the number line in order of size. Symbolically, the < and > signs are used. Again, questions relating to the analogies of temperature in meteorology and height above/below sea level are fore-grounded, before the concept of 'smaller than' or 'larger than' is connected to the symbols, all in worked examples. Research on multiple representations, case-based reasoning and analogical reasoning has demonstrated the important role of worked-out examples and concrete analogies in helping students to improve their problem-solving skills, but perhaps more significantly the importance of consistency in terms of connecting representations (analogies- pictorial/number line-symbolic) (Mayer, 1987). More interestingly, in these three parts of the chapter on negative numbers, the number line was used consistently, to further the concept from one part to the next. In addition, the same analogies to represent negative numbers were used.

Conclusions

The review of the literature on mathematical tasks in textbooks and on connections, together with the analysis of English, French and German textbooks, has deepened our understanding of the importance of connections in tasks to support understanding. When we tried to exemplify the ideas from the literature with examples from the textbooks, we found that good examples were not easy to come by. Further, we found that KM 7[2] in particular lacked any good examples at all.

Mathematical tasks can and should be seen as a process that can potentially help to enhance mathematical understanding rather than simply a vehicle for content. Thus, the nature of the mathematical task becomes crucial since each task may induce a significantly different disposition towards the mathematics. It is argued that mathematical tasks play an important part in the mathematics education process, and that mathematical tasks may enhance or impede pupil understanding, both in the classroom, as well as in a wider sense in terms of pupils' perception of what mathematics is and in which ways it is practised.

More particularly, we argue that particular tasks provide different (mis)representations of mathematics for their students in school textbooks, in particular with respect to connectivity. Despite perhaps gaining proficiency at certain kinds of procedures and tasks, some students may well have gained at best a fragmented sense of the mathematics and understood few if any connections that tie together the procedures they had studied in textbooks. It can be argued that through these disconnected activities students are likely to develop perspectives on mathematics that may impede them in their use and acquisition of other mathematical knowledge. Whereas through some tasks students are inundated with skills, procedures and disconnected mathematical knowledge, in others students are allowed to develop an appreciation of its interconnectedness and generalisable nature.

Although it would be tempting to argue that teachers mediate textbooks to ensure that tasks are appropriate and challenging, to ensure that worked examples do indeed model solution steps to multi-step examples, and to help students make connections between ideas, it is not clear from the evidence available that this is happening. We therefore have to ensure that tasks in textbooks attend to those features to ensure that students are offered appropriate kinds of learning opportunities. Indeed, if we allow textbooks to remain restrictive in what they offer, students' internal representations are in danger of remaining limited and their understanding may remain deficient. This, in turn, alerts us to the need for textbooks and teachers to make connections explicit and to support the making of connection through multiple representations.

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